REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 20	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Finite Length Discrete Matched Filters	5. TYPE OF REPORT & PERIOD COVERED: Technical Report
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Andrew B. Martinez and John B. Thomas	8. CONTRACT OR GRANT NUMBER(s): N00014-81-K-0146
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Information Sciences and Systems Laboratory, Dept. of Electrical Eng. & Computer Sci., Princeton University, Princeton, NJ 08544	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR SRO-103
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research (Code 411SP), Department of the Navy, Arlington, Virginia 22217	12. REPORT DATE: January 1986
	13. NUMBER OF PAGES: 16
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

signal detection
matched filter

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Matched Filter (MF) is well known to be the linear detector that has the maximum output Signal-to-Noise Ratio (SNR). The problem of finding the minimum filter length in discrete time to achieve a certain level of performance is considered when there is some freedom in choosing a signal shape. Upper and lower bounds on the SNR are given in terms of the eigenvalues of the noise covariance matrix. Since these bounds are rather difficult to compute, looser, but easier to compute, bounds are given. Several examples are presented which illustrate the exact and approximate bounds.

REPORT NUMBER 20

FINITE LENGTH DISCRETE MATCHED FILTERS

A.B. MARTINEZ and J.B. THOMAS

INFORMATION SCIENCES AND SYSTEMS LABORATORY

Department of Electrical Engineering
Princeton University
Princeton, New Jersey 08544

JANUARY 1986

Prepared for

OFFICE OF NAVAL RESEARCH (Code 411SP)
Statistics and Probability Branch
Arlington, Virginia 22217
under Contract N00014-81-K0146
Program in Ocean Surveillance and Signal Processing

S.C. Schwartz, Principal Investigator

FINITE LENGTH DISCRETE MATCHED FILTERS

Andrew B. Martinez
Department of Electrical Engineering
Tulane University
New Orleans, Louisiana 70118

John B. Thomas
Department of Electrical Engineering
Princeton University
Princeton, New Jersey 08540

Abstract

The Matched Filter (MF) is well known to be the linear detector that has the maximum output Signal-to-Noise Ratio (SNR). The problem of finding the minimum filter length in discrete time to achieve a certain level of performance is considered when there is some freedom in choosing a signal shape. Upper and lower bounds on the SNR are given in terms of the eigenvalues of the noise covariance matrix. Since these bounds are rather difficult to compute, looser, but easier to compute, bounds are given. Several examples are presented which illustrate the exact and approximate bounds.

I. Introduction

The design and implementation of the Matched Filter (MF) has received considerable attention [4-8]. As a detector it has the advantage of linearity, and since it is based only on easily estimated second-order noise statistics, the MF is simple to optimize. The performance criterion, the Signal-to-Noise Ratio (SNR), is tractable, and intuitively appealing.

For a fixed signal in discrete time, Levinson [8] has presented a simple and efficient algorithm to solve the MF equation. Since the MF impulse response and the SNR are computed iteratively, the algorithm can be terminated when a filter with desired performance is found. Unfortunately, when there is some freedom in choosing a signal, the choice of signal plays an important part in optimizing the detector. Because the optimal signal of length M is a truncated version of the optimal length $M+1$ signal under only very special conditions, the Levinson algorithm must be repeated N times, and thus may

lose its computational advantage. In this paper, easily computed bounds on the performance of the MF as a function of length are found. Then, before any attempt is made to solve the MF equation, an estimate of the filter length can be found from these bounds.

II. Detection Problem

The detection problem considered in this paper is one of finding a linear detector that discriminates between an hypothesis H_0 and an alternative H_1. The decision is based on a discrete length N observation vector $\mathbf{x}$ composed under H_0 of noise $\mathbf{n}$ with density f and under H_1 of a known signal $\mathbf{s}$ in noise:

$$H_0\colon \mathbf{x} = \mathbf{n}$$
$$H_1\colon \mathbf{x} = \mathbf{n} + \mathbf{s}$$

The detector consists of a real scalar test statistic $T(\mathbf{x})$, a functional of the observation $\mathbf{x}$, which is compared to a scalar threshold to decide for H_0 or H_1.

The criterion of detector optimality used in this paper is a SNR measure often called the *deflection:*

$$\mathrm{SNR} = \frac{[E_1(T) - E_0(T)]^2}{Var_0(T)} \tag{1}$$

where E_0 and E_1 are the expectation under H_0 and H_1, and Var_0 is the variance under H_0.

It is well known that the log likelihood ratio detector for Gaussian noise is linear; that is, the matched filter. Since the detector power is a monotone increasing function of the SNR of T, the SNR is frequently used as a measure of detector performance. The SNR, outside of its intuitive appeal, is often justified by making a Gaussian assumption about $\mathbf{n}$ or applying the central limit theorem to T.

Using the MF as a detector for non-Gaussian noise is more difficult to justify. In

general, the likelihood ratio detector maximizes the SNR [1], and by a simple calculus of variations argument, maximizing the SNR (as defined above) with no restriction on the linearity of the detector can be shown to yield a linear function of the likelihood ratio. The MF is the linear filter which maximizes output SNR, but the likelihood ratio is generally nonlinear. Therefore in making a restriction of linearity on T, it is tacitly assumed that the noise is Gaussian or nearly Gaussian in the sense that the MF performs reasonably well and that any loss of optimality is compensated for by the simplicity and linearity of the MF.

Under the assumption of linearity, the test statistic $T(\mathbf{x})$ is equal to the output at time N of a linear filter with impulse response $\mathbf{h}$. As a convenience, the *pseudo-signal* is defined to be a length N vector with elements $u_i = h_{N+1-i}$, the filter impulse response in reverse order. The output SNR of the linear detector is found from Eq. (1) to be

$$\mathrm{SNR}_o = \frac{\langle \mathbf{u} \mid \mathbf{s} \rangle^2}{\langle \mathbf{u} \mid \mathbf{Ru} \rangle} \leq \lambda$$

where $<\ >$ is standard inner product notation, $\mathbf{R}$ is the noise covariance matrix, and λ is the maximum value of SNR for the optimal pseudo-signal. Cross-multiplication yields

$$L(\mathbf{u}) = \langle \mathbf{u} \mid \mathbf{s} \rangle^2 - \lambda \langle \mathbf{u} \mid \mathbf{Ru} \rangle \leq 0$$

This can be maximized in the usual way by setting its gradient equal to zero:

$$\nabla L(\mathbf{u}) = 2\langle \mathbf{u} \mid \mathbf{s} \rangle \mathbf{s} - 2\lambda \mathbf{Ru} = 0$$

On rearranging and noting that $\lambda / \langle \mathbf{u} \mid \mathbf{s} \rangle$ is a constant and can be set equal to unity without loss of generality, the result is the well known MF equation:

$$\mathbf{s} = (\lambda / \langle \mathbf{u} \mid \mathbf{s} \rangle)\mathbf{Ru} = \mathbf{Ru} \tag{2}$$

The solution of Eq. (2) is the pseudo signal of the MF:

$$\mathbf{u} = \mathbf{R}^{-1}\mathbf{s}$$

with output SNR given by

$$\mathrm{SNR}_o = \langle \mathbf{u} \mid \mathbf{s} \rangle = \langle \mathbf{s} \mid \mathbf{R}^{-1}\mathbf{s} \rangle$$

In discrete time and with a fixed signal, the MF matrix equation can be solved quite efficiently using the Levinson algorithm. In continuous time, the classical method of solution is to use spectral factorization to solve the equation on an infinite interval; this (possibly) non-causal solution is then projected onto a causal space [3]. In discrete time, there is a parallel spectral approach using the eigenvectors and values of **R**.

III. Signal Selection and Bounds on the SNR

It is well known that the MF is the linear filter with the maximum SNR_o for a given signal in noise. In addition for non-white noise, the SNR_o of the MF can be maximized by proper choice of signal shape. Because of this, for signals of constant energy, the SNR_o of the MF has a range of possible values.

Since the $N \times N$ covariance matrix **R** is positive definite and Hermitian, it has positive, real eigenvalues:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$$

and a corresponding set of orthonormal eigenvectors:

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$$

The matrix **R** can be diagonalized:

$$\mathbf{R} = \mathbf{E} \Lambda \mathbf{E}^{-1}$$

where **E** is the eigenvector matrix:

$$\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N]$$

and Λ is a matrix of eigenvalues:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_N \end{bmatrix}$$

Likewise $\mathbf{R}^{-1}$ has the diagonal form:

$$\mathbf{R}^{-1} = \mathbf{E} \Lambda^{-1} \mathbf{E}^{-1}$$

where Λ^{-1} has as its diagonal elements the eigenvalues of $\mathbf{R}^{-1}$:

$$\frac{1}{\lambda_1} \geq \frac{1}{\lambda_2} \geq \cdots \geq \frac{1}{\lambda_N}$$

Thus the MF equation has the solution:

$$\mathbf{u} = \mathbf{E}\Lambda^{-1}\mathbf{E}^{-1}\mathbf{s} \tag{3}$$

The signal $\mathbf{s}$ can be expanded in terms of the eigenvectors:

$$\mathbf{s} = \mathbf{E}\mathbf{c} \tag{4}$$

where $\mathbf{c}$ is the coordinate of $\mathbf{s}$ under the basis formed by the orthonormal eigenvectors of $\mathbf{R}$. From Eqs. (3) and (4) the pseudo-signal is

$$\mathbf{u} = \mathbf{E}\Lambda^{-1}\mathbf{c} \tag{5}$$

The SNR_o of the MF becomes

$$\mathrm{SNR}_o = \mathbf{c}^T \Lambda^{-1}\mathbf{c}$$

If the signal is chosen to be in the eigenspace of the ith eigenvector ($\mathbf{s} = k\,\mathbf{e}_i$), then the MF is a simple correlator ($\mathbf{u} = \mathbf{s}$) and

$$\mathrm{SNR}_o = k/\lambda_i$$

The Rayleigh quotient theorem [2] states that

$$\frac{1}{\lambda_N} \leq \frac{<\mathbf{s}\,|\,\mathbf{R}^{-1}\mathbf{s}>}{||\mathbf{s}||^2} \leq \frac{1}{\lambda_1}$$

where the upper and lower bounds are achieved for a signals in the eigenspace of $\mathbf{e}_1$ and $\mathbf{e}_N$ respectively. Thus the SNR of the MF is bounded:

$$\frac{||\mathbf{s}||^2}{\lambda_N} \leq \mathrm{SNR}_o \leq \frac{||\mathbf{s}||^2}{\lambda_1} \tag{6}$$

The best choice of signal is $\mathbf{e}_1$, the eigenvector of $\mathbf{R}$ with the smallest eigenvalue. This is equivalent to putting the signal in that part of the spectrum of $\mathbf{R}$ where the noise has the smallest magnitude.

Grettenberg [7] has taken the logical step of using M eigenvectors as an M character alphabet of signals. By choosing the eigenvectors of $\mathbf{R}$ corresponding to the smallest M eigenvalues, not only is the set orthogonal, but it achieves the greatest minimum

SNR$_o$ of any such M character set. This also has an advantage of simplicity, since, when the signal is chosen to be an eigenvector of $\mathbf{R}$, from Eq. (9) the pseudo-signal equals the signal, and the MF reduces to a simple correlator.

A minimax strategy is used by Turin [5] to find the worst-case noise, and the corresponding best signals in continuous time. He shows that the best signal spectrum should consist of the noise spectral components with the smallest magnitude. As a consequence, the worst spectra and the best signal both have flat spectra.

IV. Levinson Algorithm and Optimal Signal Selection

For the Levinson algorithm to produce the s-optimal MF on each iteration, the length N optimal eigenvector $\mathbf{e}^{(N)}$ has to be a truncated version of the length $N+1$ eigenvector $\mathbf{e}^{(N+1)}$.

$$\mathbf{e}^{(N+1)} = \begin{bmatrix} \mathbf{e}^{(N)} \\ e_{N+1} \end{bmatrix}$$

Let $\mathbf{R}^{(N+1)}$ be the $(N+1)\times(N+1)$ covariance matrix with elements $r_{|i-j|}$, then

$$\mathbf{R}^{(N+1)}\mathbf{e}^{(N+1)} = \lambda^{(N+1)}\,\mathbf{e}^{(N+1)}$$

where $\lambda^{(N+1)}$ is the eigenvector corresponding to the eigenvector $\mathbf{e}^{(N+1)}$. Noting that the $N \times N$ minor of the covariance matrix $R^{(N+1)}$ is $R^{(N)}$:

$$\mathbf{R}^{(N+1)}\mathbf{e}^{(N+1)} = \lambda^{(N+1)}\mathbf{e}^{(N+1)} = \begin{bmatrix} \mathbf{R}^{(N)} & \mathbf{r}^{(N)} \\ (\mathbf{r}^{(N)})^T & r_0 \end{bmatrix} \begin{bmatrix} \mathbf{e}^{(N)} \\ e_{N+1} \end{bmatrix} = \begin{bmatrix} \lambda^{(N)}\mathbf{e}^{(N)} \\ (\mathbf{r}^{(N)})^T\mathbf{e}^{(N)} \end{bmatrix} + e_{N+1}\begin{bmatrix} \mathbf{r}^{(N)} \\ r_0 \end{bmatrix}$$

where

$$\mathbf{r}^{(N)} = [r_N \quad r_{N-1} \quad \cdots \quad r_1]^T$$

For $\mathbf{e}^{(N)}$ to be a truncated version of $\mathbf{e}^{(N+1)}$, then for all $N \geq 1$:

$$(\lambda^{(N+1)} - \lambda^{(N)})\mathbf{e}^{(N)} = e_{N+1}\mathbf{r}^{(N)}$$

Solving this equation iteratively yields permissible autocovariance sequences. Let r_p be the first nonzero term in the covariance sequence after r_0. Then every pth term with

index less than L is nonzero, and the rest are zero. All nonzero terms have the same magnitude with alternating or constant sign. The covariance sequences have the form:

$$r_{ip+k} = \begin{cases} r_p\,[sgn\,(r_p)]^{i-1} & k = 0 \\ 0 & k = 1, 2, \ldots, p-1 \quad \text{or} \quad ip+k \geq L \end{cases}$$

where $i \geq 0$, $0 \leq k \leq p-1$, $p \geq 0$, and $L \geq 0$. As a special case, if $L \to 0$ or $p \to \infty$ the result is white noise, $r_i = 0$ for all $i \geq 1$.

This places a severe restriction on the noise autocovariance sequence, and places corresponding limits on the utility of the Levinson algorithm for this particular problem.

V. Approximate Bounds on SNR

It is impractical to find a suitable filter length N from the bounds in Eq. (6) since they require knowledge of the eigenvalues of each $M \times M$ minor of **R**. Looser but easier to compute bounds can be found.

The equivalent rectangular time duration ΔT of the noise autocovariance is introduced as a rough measure of correlation [3]:

$$\Delta T = \sum_{i=-\infty}^{\infty} \frac{|\, r_i \,|}{\sigma^2}$$

The largest eigenvalue of **R**, denoted by λ_N, is well known to be the smallest norm of **R**, thus using another norm:

$$\lambda_N \leq \,||\mathbf{R}||_\infty = \max_i \sum_j r_{ij} \leq \sigma^2 \Delta T$$

This yields the looser bound:

$$\frac{||\mathbf{s}||^2}{\sigma^2 \Delta T} \leq \frac{||\mathbf{s}||^2}{\max\limits_i \sum\limits_j r_{ij}} \leq \frac{||\mathbf{s}||^2}{\lambda_N} \leq \mathrm{SNR}_o$$

An upper bound can be found. The condition number K of a matrix is defined as:

$$K = \lambda_{\max}/\lambda_{\min} = \,||\mathbf{R}^{-1}||\ \ ||\mathbf{R}||$$

then from Eq. (6)

$$SNR \leq ||\mathbf{s}||^2 K/\lambda_{\max}$$

The trace of **R** equals the sum of its eigenvalues:

$$tr(\mathbf{R}) = N\sigma^2 = \sum_{i=1}^{N} \lambda_i$$

therefore

$$\sigma^2 \leq \lambda_{\max} \leq N\sigma^2$$

and so

$$\frac{||\mathbf{s}||^2}{\sigma^2 \Delta T} \leq \mathrm{SNR}_o \leq \frac{||\mathbf{s}||^2 K}{\sigma^2} \tag{7}$$

Since the input *SNR* is given by

$$\mathrm{SNR}_i = \frac{||\mathbf{s}||^2}{N\sigma^2}$$

the improvement in *SNR* of the MF is given by

$$\frac{N}{\Delta T} \leq \frac{\mathrm{SNR}_o}{\mathrm{SNR}_i} = \mathrm{SNR}_{MF} \leq NK \tag{8}$$

VI. Examples

As a first example, consider white noise with an $N \times N$ covariance matrix $\mathbf{R} = \sigma^2\mathbf{I}$. The covariance matrix has an Nth order eigenvalue σ^2 making the upper and lower bounds in Eq. (6) equal, and the choice of signal arbitrary. Other considerations, such as a ceiling on transmitted signal strength, may still make the spreading of signal energy in time desirable.

For $N = 3$, the autocorrelation sequence is given by

$$\mathbf{r} = [1 \;\; r_1 \;\; r_2]^T$$

For **R** to be positive definite, the values that r_1 and r_2 can take are restricted:

$$|r_1| \leq 1 \qquad \text{and} \qquad r_2 \leq \sqrt{(r_2+1)/2}$$

This region of the $r_1 \times r_2$ plane is shown in Figs. 1 and 2.

The difference in dB between the upper and lower SNR bounds in Eq. (6) is plotted

as contours in Fig. 1. Even for a filter this short, the signal selection is shown to be quite important.

In Fig. 2, the SNR of the MF for an alternating signal ($s_i = (-1)^i$) is shown in dB over the lower bound. The alternating signal was chosen as a suboptimal approximation to the optimal signal because of its simplicity, and similarity in shape to the optimal signal for $r_1 \geq 0$. It is readily seen to be nearly optimal in this case. Because of the symmetry of this problem, a constant signal ($s_i = 1$), chosen as a suboptimal signal for $r_i < 0$ has performance contours which are the mirror image of those in Fig. 2.

Four noise autocorrelation functions were chosen as representative -- the exponential:

$$r_i = \exp(-0.2 \mid i \mid)$$

the triangular:

$$r_i = \begin{cases} 1 - \mid i \mid /10 & \mid i \mid \leq 10 \\ 0 & \mid i \mid > 10 \end{cases}$$

the Gaussian:

$$r_i = \exp(-\pi(i/10)^2)$$

and the hyperbolic secant:

$$r_i = \mathrm{sech}(\pi i/10)$$

The exponential is the simplest member of the Markov class; the triangular function has finite support; the Gaussian correlation function has infinite support, yet has tails which fall off faster than the exponential, and the hyperbolic secant has a nearly Gaussian shape at the origin, but exponential tails.

The upper and lower bounds on the SNR_{MF} from Eq. (6) are plotted in dB versus filter length N in Figs. 3-6. Here signal selection is extremely important for all $N > 2$ and increasingly so for increasing N. Even for the length 5 filter, the difference between the best and worse-case SNR_{MF} is at least 15dB for all four cases. At $N = 20$, the

difference is at least 19dB.

The parameters of these four correlation functions were chosen so that each has an equivalent rectangular time duration of $\Delta T \approx 10$. Thus the approximate lower bound (Eq. 8) for each function is

$$N/10 \leq \mathrm{SNR}_{MF}$$

The approximate and exact lower bounds for each correlation function is shown is Figs. 7-10.

VII. Conclusions

The bounds on SNR_{MF} in Eq. (6) show the selection of the signal to be important for optimal performance of the MF. The selection of a suboptimal signal, if made intelligently, can produce nearly optimal results, and certainly the importance of signal shape should not be overlooked.

The approximate lower bound of Eq. (8) gives a simple although conservative estimate of worse case MF performance. An estimate of filter length can be made with only knowledge of the equivalent rectangular time duration ΔT.

Acknowledgement

This research is supported by the Office of Naval Research under Contract N00014-81-K-0146 and by the National Science Foundation under Grant ECS-83-17777.

References

[1]. C.W. Helstrom, *Statistical Theory of Signal Detection.* Oxford:Pergamon Press, 1968.

[2]. H.R. Schwartz, H. Rutishauser, and E. Stiefel, *Numerical Analysis of Symmetric Matrices.* New York:Prentice Hall, 1973.

[3]. J.B. Thomas, *Introduction to Statistical Communication Theory.* New York:Wiley, 1969.

[4]. G.L. Turin, "An Introduction to Matched Filters," *IRE Trans. on Information Theory,* Vol. IT-6, June 1960, pp. 311-329.

[5]. G.L. Turin, "Minimax Strategies for Matched Filter Detection," *IEEE Trans. on Communications,* Vol. COM-23, No. 11, Nov. 1975, pp. 1370-1371.

[6]. G.L. Turin, "An Introduction to Digital Matched Filters," *Proc. of IEEE,* Vol. 64, No. 7, July 1976, pp. 1092-1112.

[7]. T.L. Grettenberg, "Signal Selection in Communication and Radar Systems" *IEEE Trans. on Information Theory,* Vol. IT-9, No. 4, Oct. 1963, pp. 265-275.

[8]. N. Levinson, "The Wiener RMS (Root Mean Square) Error Criterion in Filter Design and Prediction," *Journal of Mathematics and Physics,* Vol. XXV, No. 4, Jan. 1947, pp. 261-278.

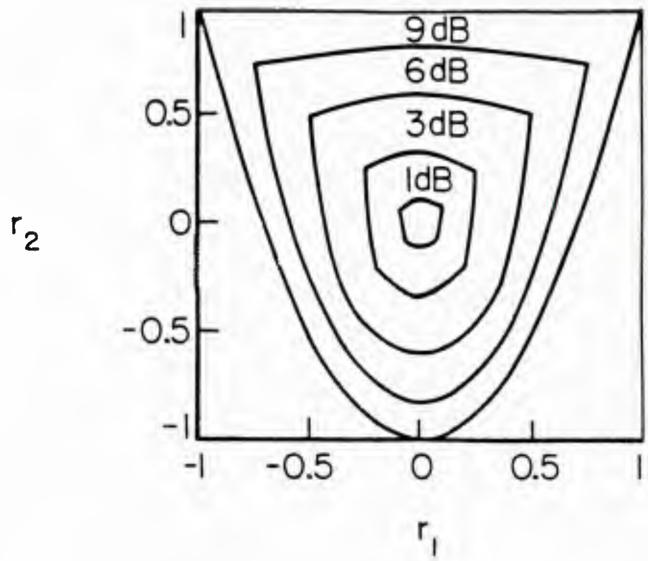

Figure 1 – Contours of SNR_{MF} upper bound for N = 3 in $r_1 \times r_2$ plane.

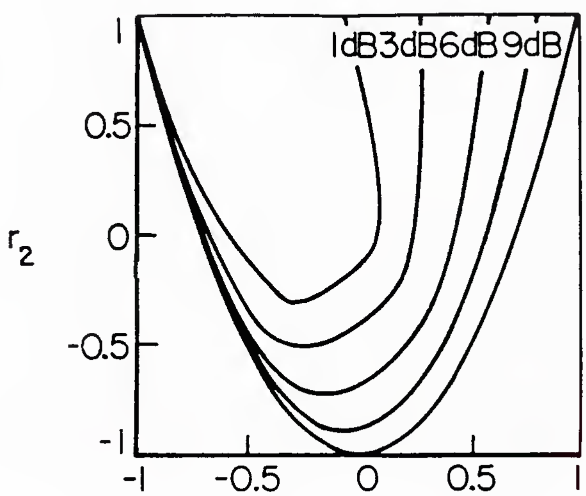

Figure 2 – Contours of SNR_{MF} for alternating signal in $r_1 \times r_2$ plane.

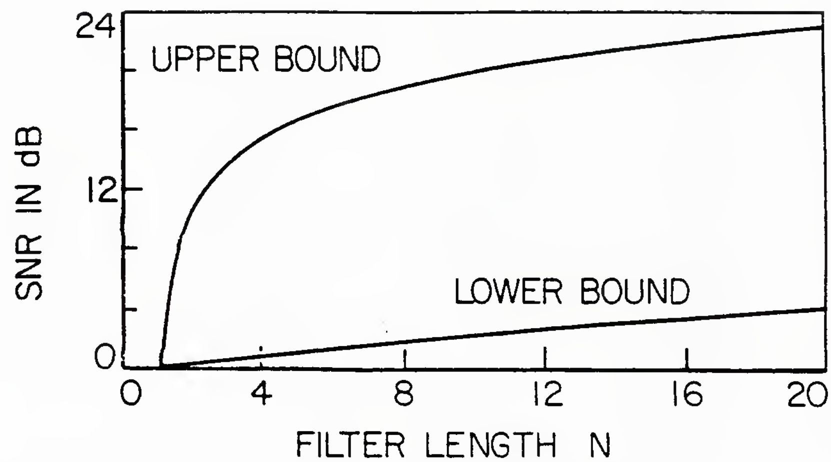

Figure 3 - Upper and lower bounds on SNR_{MF} for exponential correlation.

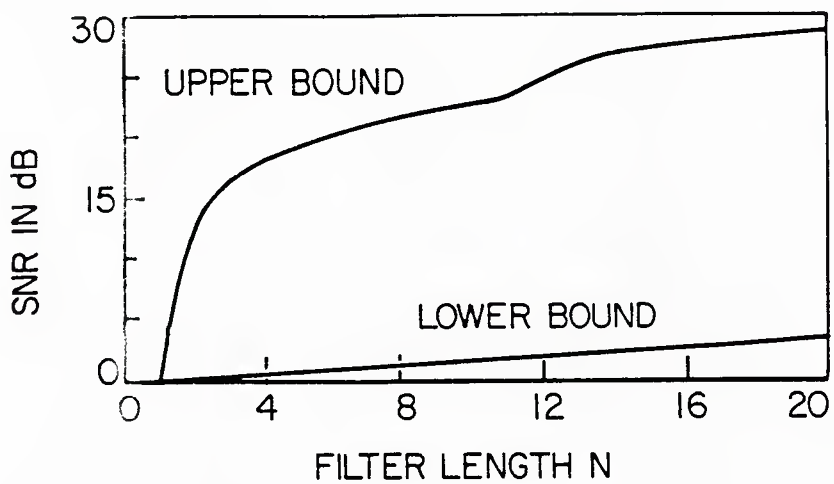

Figure 4 - Upper and lower bounds on SNR_{MF} for triangular correlation.

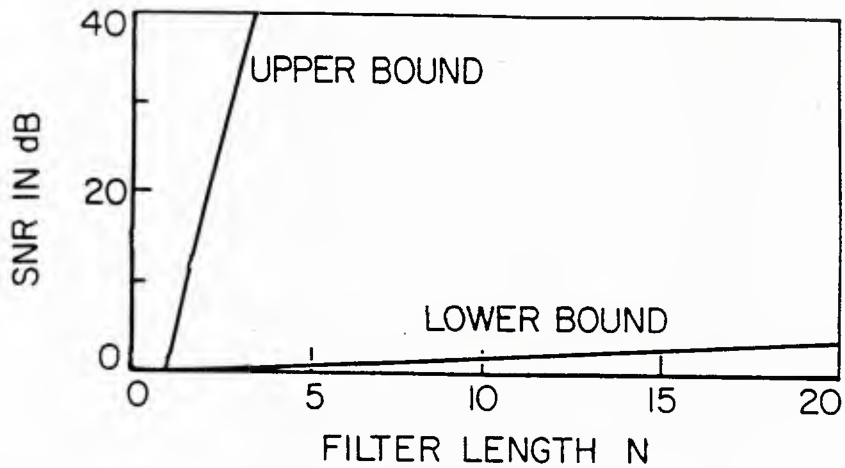

Figure 5 - Upper and lower bounds on SNR_{MF} for Gaussian correlation.

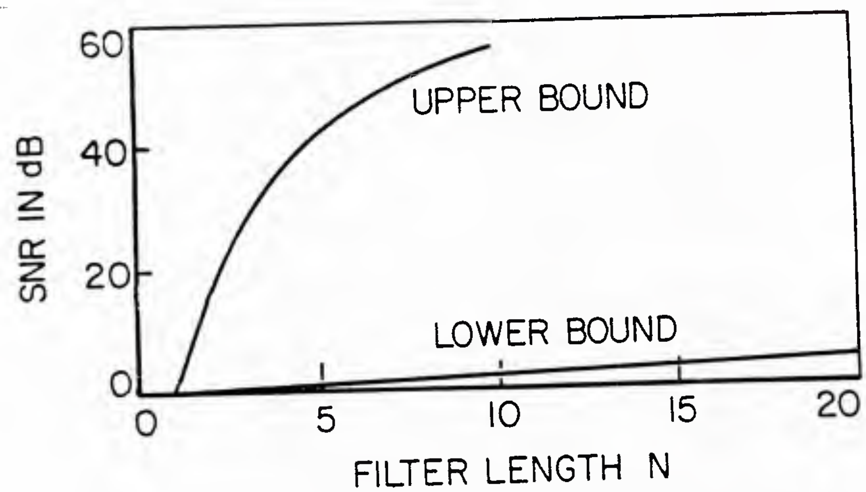

Figure 6 - Upper and lower bounds on SNR_{MF} for hyperbolic secant correlation.

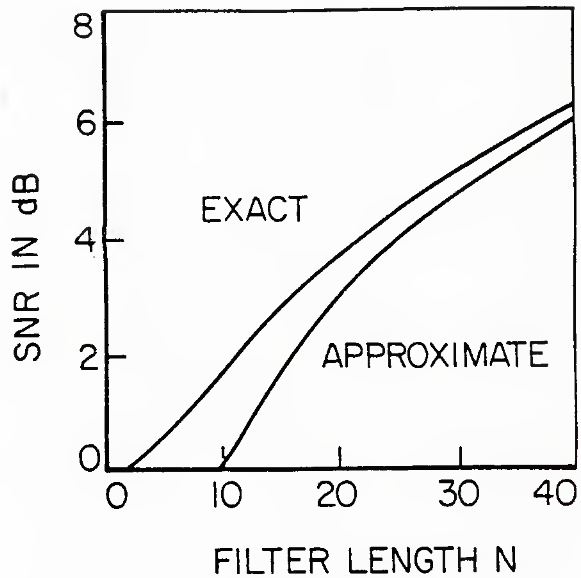

Figure 7 - Approximate and exact lower bound for exponential correlation.

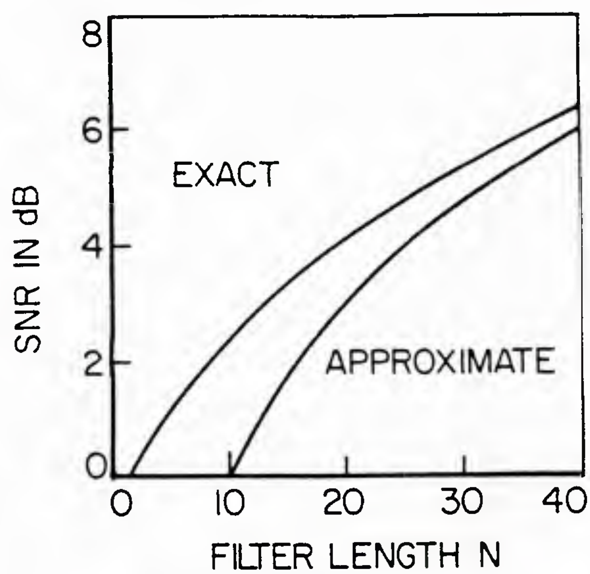

Figure 8 - Approximate and exact lower bound for triangular correlation.

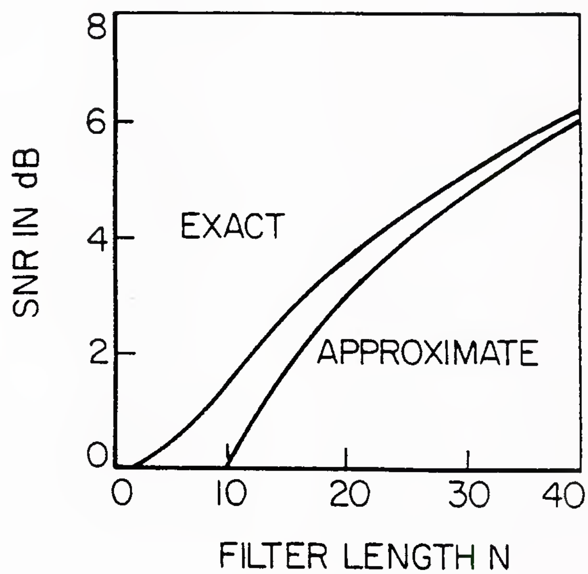

Figure 9 - Approximate and exact lower bound for Gaussian correlation.

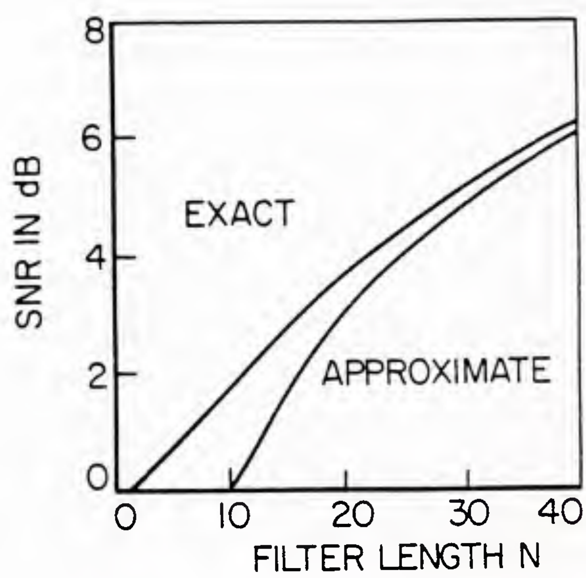

Figure 10 - Approximate and exact lower bound for hyperbolic secant correlation.